

The classical bosonic string

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Outline

1 Introduction

- Preliminaries
- Particles and beyond

2 Basic results in strings

- Action functional
- Equations of motion and constraints

3 Further results

- Fourier expansion of the solution
- Momenta, mass and more
- The Virasoro algebra



Conventions

- Units choice: $\hbar = c = 1$.
- Any metric we will use has the pseudo-Euclidean signature — it has one negative and $d - 1$ positive eigenvalues, corresponding to time-like and space-like coordinates, respectively.
- The metrics are usually dependent on the space-time point (curved space-time), but we will omit (x) in $m_{\mu\nu}(x)$.
- For any tensor $\mathfrak{T}^{\mu\nu}$, its core symbol (without indices) stands for the absolute value of its determinant:

$$\mathfrak{T} \equiv |\det \mathfrak{T}_{\mu\nu}|$$

A few shorthands

- $A \cdot B \equiv A_\mu B^\mu \equiv m_{\mu\nu} A^\mu B^\nu$, where $m^{\mu\nu}$ is the relevant metric,
- $X^2 \equiv X \cdot X$,
- $\dot{X} \equiv \frac{dX}{d\tau}$, where τ is time-like,
- $\delta_m \equiv \delta_{0m}$ — one-index Kronecker delta.

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Relativistic particle

Recollection

- Relativistic action of a particle:

$$S = -m \int ds^2, \text{ where } ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$$

- Transforming to integral over $d\tau$:

$$S = -m \int d\tau \sqrt{-\dot{x}^2}, \text{ where } \dot{x}^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

- Obviously invariant under $\tau \rightarrow \tilde{\tau}$ reparametrisations.
- It does not apply to mass-less particles.

Relativistic particle

Mass-less form

- Introducing the auxiliary coordinate, 'einbein':

$$S = \frac{1}{2} \int (e^{-1} \dot{x}^2 - em^2) d\tau$$

- Einbein equation of motion (typical Euler-Lagrange procedure):

$$\dot{x}^2 + e^2 m^2 = 0$$

- As easy as recovering the previous form of S . Just solve the equation to $e = \frac{\sqrt{-\dot{x}^2}}{m}$, substitute in S above and simplify.

Relativistic particle

Gauge choice, momentum

- Invariance wrt infinitesimal transformations:

$$\delta x = \xi \dot{x}, \quad \delta e = \frac{d(\xi e)}{d\tau}$$

- We use gauge symmetry to ensure $e = m^{-2}$. We obtain the following momentum, Hamiltonian and equations of motion:

$$p_\mu = m^2 \dot{x}_\mu$$

$$H = \frac{1}{2}(1 + m^{-2} p^2)$$

$$\dot{p}_\mu = 0, \quad \dot{x}_\mu = m^{-2} p_\mu$$

- Non-trivial fact: e EOM is still a constraint — mass-shell condition of the form:

$$p^2 = -m^2$$

Relativistic particle

Generalisation

- 0-dimensional particles are just a matter of our custom!
- We can consider objects such as strings, membranes, or their n -dimensional equivalents.
- $n + 1$ -dimensional world-manifold is embedded in D -dimensional space-time, thus $D > n$.
- Generalised action has the following (Polyakov) form:

$$S = \frac{T}{2} \int d^{n+1}\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu$$

$h^{\alpha\beta}$ is a metric describing the world-manifold, $g^{\mu\nu}$ — the space-time metric, $\sigma^0 \equiv \tau$ is time-like.

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$$S = \frac{T}{2} \int d^{n+1}\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu$$

Generalised string action

- Invariance: $\sqrt{-h} d^{n+1}\sigma$ is manifestly invariant, the rest is a scalar.
- Reparametrisations: out of $\frac{1}{2}(n+1)(n+2)$ independent components we can gauge $n+1$ away through reparametrisations. $\frac{1}{2}n(n+1)$ components still remain.
- Weyl scaling allows one more component to be fixed:

$$\begin{aligned} h_{\alpha\beta} &\rightarrow \Lambda(\sigma) h_{\alpha\beta} \\ \sqrt{-h} h^{\alpha\beta} &\rightarrow \Lambda^{\frac{1}{2}(n-1)} \sqrt{-h} h^{\alpha\beta} \end{aligned}$$

- Thus for $n=1$ (strings) all the $h^{\alpha\beta}$ variability can be gauged away.

Why only strings?

- By power counting, the action is not renormalisable for $n > 1$.
- Thus only the strings are usually considered.
- String tension T has dimensions of $L^{-2} = M^2$. It is related to the Regge slope as follows:

$$T = (2\pi\alpha')^{-1}$$

- In Minkowski space, the action reduces to:

$$S = \frac{T}{2} \int_M d^2\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) \partial_\alpha X^\mu \partial_\beta X_\mu$$

- For convenience, $0 \leq \sigma \leq \pi$.

Additional terms in the action

- ‘Cosmological constant’ term:

$$S_1 = \lambda \int d^2\sigma \sqrt{h}$$

Not considered — not invariant under Weyl scaling ($\sqrt{h}h^{\alpha\beta}$ is invariant, but sole \sqrt{h} is not). Thus either $h_{\alpha\beta}$ or λ is 0.

- Curvature term:

$$S_2 = \frac{1}{2\pi} \int d^2\sigma \sqrt{h} R^{(2)}(h)$$

$R^{(2)}(h)$ is the scalar curvature of the world sheet, significant in string interactions.

- The term $\sqrt{h}R^{(2)}$ is a total derivative — doesn’t contribute to the results.



Action symmetries

- Reparametrisation generalised:

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$

$$\delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}$$

$$\delta(\sqrt{h}) = \partial_\alpha(\xi^\alpha \sqrt{h})$$

- Weyl scaling:

$$\delta h^{\alpha\beta} = \Lambda h^{\alpha\beta}$$

- Poincaré (full relativistic) invariance ($a_{\mu\nu} = -a_{\nu\mu}$):

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu$$

$$\delta h^{\alpha\beta} = 0$$

Energy–momentum tensor

- We can obtain the e–m tensor by varying S wrt the metric:

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}}$$

- The result is as follows:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X_\mu$$

- Under Weyl scaling $h^{\alpha\beta} \rightarrow \Lambda h^{\alpha\beta}$ the trace transforms:

$$h^{\alpha\beta} T_{\alpha\beta} \rightarrow \Lambda h^{\alpha\beta} T_{\alpha\beta}$$

- This implies $T^\alpha_\alpha = 0$.

Energy–momentum tensor (cont.)

- Field equation $0 = \delta S / \delta h^{\alpha\beta} \sim T_{\alpha\beta}$.
- With $G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$, the above gives:

$$G_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} h^{\alpha'\beta'} G_{\alpha'\beta'}$$

$$G = \frac{1}{4} h (h^{\alpha\beta} G_{\alpha\beta})^2$$

$$S = \frac{1}{2} \int_{\Sigma} d^2\sigma \sqrt{h} h^{\alpha\beta} G_{\alpha\beta} = \int_{\Sigma} d^2\sigma \sqrt{G}$$

- The latter is simply the area of the world-tube.

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Equations of motion

Gauge choice

From now on, we assume that the gauge and Weyl scaling is chosen so that the world-manifold metric is locally flat, thus equal to Minkowski metric

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Action in simplified form:

$$S = \frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X$$

Equations of motion (cont.)

- Typical Euler-Lagrange procedure (where $f' \equiv \frac{\partial f}{\partial \sigma}$):

$$\frac{\partial L}{\partial X^\mu} = 0$$

$$\frac{\partial L}{\partial(\partial_\alpha X^\mu)} = -T \partial^\alpha X_\mu$$

$$\square X^\mu \equiv X''^\mu - \ddot{X}^\mu = \partial^\alpha \partial_\alpha X^\mu = 0$$

Boundary conditions

- Invariance under general variation: $X^\mu \rightarrow X^\mu + \delta X^\mu$.
- Action varies as follows (omission of terms super-linear in δ):

$$S \rightarrow S - T \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta \delta X_\mu$$

- After integration by parts we obtain (after omission of terms, which vanish after integration over $d\tau$):

$$-T \int d\tau X'_\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

- Vanishing requirement gives boundary conditions:
 - $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$ (periodicity) for closed strings,
 - $X'^\mu = 0$ at $\sigma = 0, \pi$ (von Neumann condition) for open strings.

D-branes

Digression

- We could choose Dirichlet boundary conditions, i.e.:

$$\delta X^\mu = 0 \text{ at } \sigma = 0, \pi$$

- This assumption violates Poincaré (translational) invariance — unless the endpoints of a string are attached to a moving object, called in general a D-brane.
- Energy flow through the endpoint of a string is nonzero.
- Some physical entity (D-brane) has to be a source of such flow (energy conservation law).



General solution

- Usual solution as a sum of arbitrary functions (left- and right-moving modes):

$$X^\mu(\sigma, \tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$$
$$\sigma^\pm = \tau \pm \sigma$$

- Conjugate derivatives and Minkowski metric tensor in light-cone coordinates:

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$$
$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}$$
$$\eta_{++} = \eta_{--} = 0$$

Constraint equations

- Expanding $T_{\alpha\beta} = 0$ and in ordinary and light-cone coordinates:

$$T_{10} = T_{01} = \dot{X} \cdot X' = 0$$

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X \cdot \partial_+ X$$

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = \partial_- X \cdot \partial_- X$$

$$T_{+-} = T_{-+} = 0$$

The last follows from tracelessness.

- $T_{\pm\pm} = 0$ implies following statements:

$$\dot{X}_R^2 = \dot{X}_L^2 = 0$$

Conformal invariance

- Energy–momentum conservation law in conformally invariant case:

$$\partial_- T_{++} = 0$$

- Infinite set of conserved quantities — for any function f of σ^+ , current fT_{++} is conserved.
- Setting $h^{\alpha\beta} = \eta^{\alpha\beta}$ still allows gauge change, consider

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}$$

- Then in light-cone coordinates, the $\xi^\pm = \xi^0 \pm \xi^1$ may be arbitrary functions of σ^\pm .
- This residual symmetry is generated by:

$$V^\pm = \xi^\pm(\sigma^\pm) \frac{\partial}{\partial \sigma^\pm}$$

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Decomposition into Fourier series

Closed strings

- Periodicity conditions: $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$.
- Corresponding decomposition into Fourier series:

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau - \sigma) + \frac{i l}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau + \sigma) + \frac{i l}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}$$

- Fundamental length:

$$l = \sqrt{2\alpha'} = 1/\sqrt{\pi T}$$

Decomposition into Fourier series

Closed strings (cont.)

- x^μ — centre of mass position, p^μ — momentum.
- Terms linear in σ cancel out in X^μ — satisfaction of the boundary condition.
- Reality requirement:

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$$

- Derivatives of the solution:

$$\partial_- X_R^\mu = \dot{X}_R^\mu = l \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-2in(\tau-\sigma)}$$

$$\partial_+ X_L^\mu = \dot{X}_L^\mu = l \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}$$

Decomposition into Fourier series

Closed strings — Poisson brackets

- At fixed τ we obtain:

$$[X^\mu(\sigma), X^\nu(\sigma')]_{\text{P.B.}} = [\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')]_{\text{P.B.}} = 0$$

$$[\dot{X}^\mu(\sigma), X^\nu(\sigma')]_{\text{P.B.}} = T^{-1} \delta(\sigma - \sigma') \eta^{\mu\nu}$$

- After expansion insertion we get:

$$[\alpha_m^\mu, \alpha_n^\nu]_{\text{P.B.}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = im \delta_{m+n} \eta^{\mu\nu}$$

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = 0$$

- It remains valid with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} l p^\mu$, and implies

$$[p^\mu, x^\nu]_{\text{P.B.}} = \eta^{\mu\nu}$$

Decomposition into Fourier series

Open strings

- Boundary condition: $X'^{\mu}|_{\sigma=0,\pi} = 0$.
- Standing wave solution:

$$X^{\mu} = x^{\mu} + l^2 p^{\mu} \tau + il \sum_{n \neq 0} \alpha_n^{\mu} e^{-in\tau} \cos n\sigma$$

$$2\partial_{\pm} X^{\mu} = \dot{X}^{\mu} \pm X'^{\mu} = l \sum_{n=-\infty}^{\infty} \alpha_n^{\mu} e^{-in(\tau \pm \sigma)}, \text{ where } \alpha_0^{\mu} \equiv lp^{\mu}$$

- Contrary to the closed case, left- and right-moving modes are dependent.

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Noether method

- Noether theorem: current J_α associated with transformation $\phi(\sigma) \rightarrow \phi(\sigma) + \epsilon \delta\phi(\sigma)$ is conserved.
- Allowing variability of ϵ — $\phi(\sigma) \rightarrow \phi(\sigma) + \epsilon(\sigma)\delta\phi(\sigma)$ leads to non-invariant action (proportionally to ϵ derivative):

$$\delta S = \int d^2\sigma J_\alpha \partial^\alpha \epsilon$$

- EOM can be satisfied only if $\delta S = 0$ for all ϵ , which is true when

$$\partial^\alpha J_\alpha = 0 \text{ (obtained through integration by parts)}$$

Linear and angular momentum

Currents associated with Poincaré transformation:

- Translation $\delta X^\mu = b^\mu$ — density of linear momentum:

$$\delta S = T \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha b_\mu \Rightarrow P_\alpha^\mu = T \partial_\alpha X^\mu$$

- Rotation $\delta X^\mu = a^\mu{}_\nu X^\nu$ — density of angular momentum

$$\delta S = -T \int d^2\sigma \partial^\alpha a_{\mu\nu} (X^\nu \partial_\alpha X^\mu) \Rightarrow$$

$$J_\alpha^{\mu\nu} = T (X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu)$$

Linear and angular momentum (cont.)

- Conservation laws:

$$\partial_\alpha P^{\alpha\mu} = \partial_\alpha J^{\alpha\mu\nu} = 0$$

- Amount of momentum flowing across a line segment:

$$dP^\mu = P^\mu_\tau d\sigma + P^\mu_\sigma d\tau$$

- No momentum flows through the endpoints of an open string:

$$\left. \frac{dP^\mu}{d\tau} \right|_{\sigma=0,\pi} = P^\mu_\sigma|_{\sigma=0,\pi} = T \partial_\sigma X^\mu|_{\sigma=0,\pi}$$

which vanishes under von Neumann conditions.

Linear and angular momentum (cont.)

- Total linear momentum — Fourier terms with $n \neq 0$ vanish after integration:

$$P^\mu = T \int_0^\pi d\sigma \left. \frac{dX^\mu(\sigma)}{d\tau} \right|_{\tau=0} = \pi T (l\alpha_0^\mu + l\tilde{\alpha}_0^\mu) = p^\mu$$

- Total angular momentum:

$$J^{\mu\nu} = T \int_0^\pi d\sigma \left(X^\mu \frac{dX^\nu}{d\tau} - X^\nu \frac{dX^\mu}{d\tau} \right)$$

Angular momentum (cont.)

- After inserting the expansion we obtain:

$$J^{\mu\nu} = I^{\mu\nu} + E^{\mu\nu} \quad \text{for open strings}$$

$$J^{\mu\nu} = I^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \quad \text{for closed strings}$$

where we used:

$$I^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu \quad E^{\mu\nu} \equiv -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu)$$

and an equivalent expression for $\tilde{E}^{\mu\nu}$ in terms of $\tilde{\alpha}_n^\mu$.

Examples

Rotating loop

- Let's consider a closed string on the xy plane. At $\tau = t = 0$

$$x = R \cos 2\sigma, \quad y = R \sin 2\sigma$$

- It is easy to see** that EOM and constraints can be satisfied if

$$t = 2R\tau \text{ near } \tau = t = 0$$

- We don't know — maybe the authors used some inconsistent (or really advanced!) approximation in $t \rightarrow 0$.
- Anyway, if we believe them, clearly

$$E = P^0 = T \int_0^\pi d\sigma 2R = 2\pi RT$$

so the T is really the string energy per length — that is, tension.

Regge trajectory

Digression

- Tullio Regge studied amplitudes of scattering in quantum mechanics.
- He assumed that quantum number l may be complex in general, not only integer.
- Thus the amplitudes had to be analytically continued over the complex plane.
- The function had several poles, which 'travelled' when scattering energy was varied — this 'travel' is described by the Regge trajectory.
- When this trajectory is (assumed to be) linear, it is described with the slope and intercept, $\alpha(s) = \alpha' s + \alpha(0)$.
- One of the discovered properties:

$$m^2 \sim \frac{J}{\alpha'}$$



Examples

Rotating straight string

- Proof that α' is the Regge slope — maximum angular momentum per E^2 . Let us examine the following case:

$$x = A \cos \tau \cos \sigma \quad y = A \sin \tau \cos \sigma \quad t = A\tau$$

Clearly satisfies EOM and constraints. From previous results we obtain $E = P^0 = \pi AT$, $J_z = J^{12} = \pi A^2 T/2$, and

$$J \cdot E^{-2} = (2\pi T)^{-1}$$

- Observation: endpoints of an open string move at c (consequence of boundary condition and $\frac{1}{2}(\dot{X}^2 + X'^2) = 0$ constraint).

Squared mass

- String squared mass is equal to $-p_\mu p^\mu$:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \quad \text{for open strings}$$

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \quad \text{for closed strings}$$

- These are the mass-shell conditions, analogous to the one derived earlier for the particle.

Two-dimensional Hamiltonian

- The Hamiltonian of the 2-D theory:

$$H = \int_0^\pi d\sigma (\dot{X} \cdot P - L) = \frac{T}{2} \int_0^\pi d\sigma (\dot{X}^2 + X'^2)$$

- In our case(s):

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n \quad \text{for open strings}$$

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \quad \text{for closed strings}$$

Constraint Fourier series

Closed strings

- Expanding the constraints $\dot{X}_R^2 = \dot{X}_L^2 = 0$ in Fourier series for closed strings:

$$\begin{aligned}
 L_m &= \frac{T}{2} \int_0^\pi d\sigma e^{-2im\sigma} T_{--} \\
 &= \frac{T}{2} \int_0^\pi d\sigma e^{-2im\sigma} \dot{X}_R^2 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \\
 \tilde{L}_m &= \frac{T}{2} \int_0^\pi d\sigma e^{2im\sigma} T_{++} \\
 &= \frac{T}{2} \int_0^\pi d\sigma e^{2im\sigma} \dot{X}_L^2 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n
 \end{aligned}$$

- $H = L_0 + \tilde{L}_0.$

Constraint Fourier series

Open strings

- First we need to assume periodicity in σ beyond the interval $[0, \pi]$:

$$X_{R/L}(\sigma + \pi) = X_{R/L}(\sigma)$$

- Then the expansion is as follows:

$$\begin{aligned} L_m &= T \int_0^\pi d\sigma (e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}) \\ &= \frac{T}{4} \int_{-\pi}^\pi d\sigma e^{im\sigma} (\dot{X} + X')^2 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \end{aligned}$$

- $H = L_0$.

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The Virasoro algebra

- Virasoro algebra is spanned by elements $\mathcal{L}_i \in \mathbb{C}, i \in \mathbb{Z}$ and so-called central charge c , which obey the following:

$$\mathcal{L}_n + \mathcal{L}_{-n} \in \mathbb{R}$$

$$[c, \mathcal{L}_n]_{\text{P.B.}} = 0$$

$$[\mathcal{L}_m, \mathcal{L}_n]_{\text{P.B.}} = i(m-n)\mathcal{L}_{m+n} + \frac{ic}{12}(m^3 - m)\delta_{m+n}$$

- Now we'll prove that L_n satisfy these conditions for $c = 0$.

Virasoro algebra of e-m tensor modes

- From the definition of L_n :

$$[L_m, L_n]_{\text{P.B.}} = \frac{1}{4} \sum_{k,l} [\alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l]_{\text{P.B.}}$$

- Applying the P.B./commutator/etc. identity:

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$$

we obtain:

$$[L_m, L_n]_{\text{P.B.}} = \frac{i}{4} \sum_{k,l} (k\alpha_{m-k} \cdot \alpha_l \delta_{k+n-l} + k\alpha_{m-k} \cdot \alpha_{n-l} \delta_{k+l} \\ + (m-k)\alpha_l \cdot \alpha_k \delta_{m-k+n-l} + (m-k)\alpha_{n-l} \cdot \alpha_k \delta_{m-k+l})$$



Virasoro algebra of e-m tensor modes (cont.)

- After summation over δ -indices:

$$[L_m, L_n]_{\text{P.B.}} = \frac{i}{2} \sum_k k \alpha_{m-k} \cdot \alpha_{k+n} + \frac{i}{2} \sum_k (m-k) \alpha_{m-k+n} \cdot \alpha_k$$

- Finally, index change $k \rightarrow k+n$ finishes the proof:

$$[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n}$$

- The same can be proved for \tilde{L}_n .

Virasoro algebra interpretation

- Consider a circle S^1 on a plane, parametrised by $\theta \in (0, 2\pi]$.
- Generator of an infinitesimal, general transformation $\theta \rightarrow \theta + a(\theta)$:

$$D_a = ia(\theta) \frac{d}{d\theta}$$

- A complete basis of these transformations is given by:



$$D_n = ie^{in\theta} \frac{d}{d\theta}$$

- These clearly obey the Virasoro algebra conditions with $c = 0$.

Analogy to residual symmetries

- By replacing $e^{in\theta} \rightarrow \xi^\pm$ and $i \frac{d}{d\theta} \rightarrow \frac{\partial}{\partial \sigma^\pm}$ in generators of S^1 transformations, we recover the generators of residual symmetries.
- Variables σ^\pm are not a priori angular.
- To fulfil the EOM of the theory, mode expansions of strings contain $\exp in\sigma^\pm$ only for integer n .
- Thus σ^\pm may be considered angular.

Sources and further reading

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Cambridge University Press, 1987.
-  Lev L. Landau, Evgeny M. Lifshitz
Quantum Mechanics — Non-relativistic Theory
Pergamon Press, 1977.

§141 of [2] covers Regge poles.

Sources and further reading (cont.)



Wikipedia, the free encyclopedia

Articles on string-related topics:

http://en.wikipedia.org/wiki/String_theory

<http://en.wikipedia.org/wiki/D-branes>

...



Википедия, свободная энциклопедия

Article on Regge theory

http://ru.wikipedia.org/wiki/Теория_Реджэ