Torsion: what it is and how does it work

Paweł Laskoś-Grabowski

Institute for Theoretical Physics, University of Wrocław

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Outline

1. Introduction
   - Conventions
   - Einstein–Cartan theory
2. Mathisson–Papapetrou equation
   - Free spinning particle
   - Particle in gravitational field
3. Finish
   - Motivations and conclusions
   - Bibliography

Basic symbols & conventions

- $D = 4$
- $c = \hbar = 1$
- $\eta = diag(-, +, +, +)$
- Greek indices correspond to the curved space. Latin indices correspond to the tangent space.

Outline

Poincaré group

- Initially devised to be the symmetry group of Maxwell equations
- Consists of (local) translations, rotations and boosts
- Algebra consists of 10 generators: momenta and angular momenta
- Commutation relations:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \\
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\sigma} J_{\mu\rho})
\end{aligned}
$$

- We now choose it to be group of fundamental symmetries of our theory

Differential forms

- Coordinate-independent approach
- $\alpha \equiv \alpha_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$
- Wedge product: $\alpha \wedge \beta$ is a form of greater rank
- Exterior derivative $d\alpha$:

$$d\alpha = \sum_j \frac{\partial \alpha_{i_1 i_2 \ldots i_k}}{\partial x^j} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \wedge dx^j$$

Einstein–Cartan theory

- Gauge fields for Poincaré group
  - Translations P^a ➜ field of *frames* $e^a \equiv e^a_\mu \, dx^\mu$
  - Lorentz rotations $\Omega_{ab} = -\Omega_{ba}$ ➜ field of *connection* $\omega^{ab} \equiv \omega^{ab}_\mu \, dx^\mu$
- Transformations under Lorentz group element Λ

$$\omega' = \Lambda\eta\omega\eta\Lambda^T - (d\Lambda)\eta\Lambda^T$$
$$e' = \Lambda\eta e$$

- Transformations under translation ξ^a

$$\omega' = \omega$$
$$e' = e + d\xi + \omega\eta\xi$$

Curvature and torsion

- Two important two-forms

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$$
$$T^a = de^a + \omega^a{}_b \wedge e^b$$

- Bianchi identity (curvature is covariant constant)

$$\begin{aligned} dR &= dd\omega + d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega \\ &= d\omega \wedge \omega + \omega \wedge \omega \wedge \omega - \omega \wedge d\omega - \omega \wedge \omega \wedge \omega \\ &= (d\omega + \omega \wedge \omega) \wedge \omega - \omega \wedge (d\omega + \omega \wedge \omega) = R \wedge \omega - \omega \wedge R \end{aligned}$$

- Similar identity for torsion

$$\begin{aligned} dT &= dde + d(\omega \wedge e) = d\omega \wedge e - \omega \wedge de \\ &= d\omega \wedge e + \omega \wedge \omega \wedge e - \omega \wedge de - \omega \wedge \omega \wedge e \\ &= (d\omega + \omega \wedge \omega) \wedge e - \omega \wedge (de + \omega \wedge e) = R \wedge e - \omega \wedge T \end{aligned}$$

Relation to 'classical' general relativity

- e is invertible ➜ $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$
- May be proven that

$$R^a{}_b - \frac{1}{2}\eta^a_b R + \lambda \eta^a_b = \kappa T^a{}_b$$
$$T^a_{bc} = \kappa S^a_{bc} + \frac{\kappa}{D-2} S^d_{db}\eta^a_c - \frac{\kappa}{D-2} S^d_{dc}\eta^a_b$$

- Classical GR assumes also $T = 0$ and $e = const$. Everything, incl. R may then be expressed in terms of g.

Outline

Representation

- Particles represented as elements of connected Poincaré group

$$P_+^\uparrow = \mathbb{R}^4 \times L_+^\uparrow = \{(z, \Lambda)\}$$

- z — coordinates of particle location
- Λ relates to momentum and spin:
  - $p_a = m\Lambda_{a0}, m > 0$
  - $\frac{1}{2} S_{ab}\sigma^{ab} = \lambda \Lambda \sigma_{12} \Lambda^{-1} \equiv -iS$
    where $(\sigma^{ab})_{cd} = -i(\eta_c^a \eta_d^b - \eta_d^a \eta_c^b)$, $\lambda = const$
- It follows that $\frac{1}{2} S_{ab} S^{ab} = \lambda^2, p_a S^{ab} = 0$

Free lagrangian

- Classical spinning particle: $L_0 = \frac{1}{2} m \dot{\vec{x}}^2 + \lambda i \,\mathrm{Tr}(\sigma_3 s^{-1} \dot{s})$
- Relativistic analogue: $L_p = p_a \dot{z}^a + i\frac{\lambda}{2} \,\mathrm{Tr}(\sigma_{12} \Lambda^{-1} \dot{\Lambda})$
- Variation w.r.t. z ➜ obviously $\dot{p}_a = 0$
- General variation of Λ is of such form: $\delta\Lambda = i\epsilon \cdot \sigma \,\Lambda$
- Then follows $\delta\Lambda^{-1} = -i\Lambda^{-1}\,\epsilon \cdot \sigma$ — because $0 = \delta(\Lambda\Lambda^{-1}) = \delta\Lambda\,\Lambda^{-1} + \Lambda\,\delta\Lambda^{-1} = i\epsilon \cdot \sigma\,\Lambda\Lambda^{-1} + \Lambda\,\delta\Lambda^{-1}$

Free lagrangian (cont.)

Variation w.r.t. Λ

- If $k^{ab} = \dot{z}^a p^b = (z^a p^b)^{\bullet}$, then

$$\begin{aligned}\delta(p_a \dot{z}^a) &= m\dot{z}^a \delta\Lambda_{a0} = mi(\epsilon\cdot\sigma)_{ab}\Lambda^b_0 \dot{z}^a \\ &= -i(\epsilon\cdot\sigma)_{ba} k^{ab} = -i\,\mathrm{Tr}(\epsilon\cdot\sigma\, k)\end{aligned}$$

- The other term yields

$$\begin{aligned}i\frac{\lambda}{2}\delta(\mathrm{Tr}(\sigma_{12}\Lambda^{-1}\dot{\Lambda})) &= i\frac{\lambda}{2}\,\mathrm{Tr}(\sigma_{12}\cdot -i\Lambda^{-1}(\epsilon\cdot\sigma)\dot{\Lambda}) \\ &+ i\frac{\lambda}{2}\,\mathrm{Tr}(\sigma_{12}\Lambda^{-1} i(\epsilon\cdot\sigma)^{\bullet}\Lambda) \\ &+ i\frac{\lambda}{2}\,\mathrm{Tr}(\sigma_{12}\Lambda^{-1} i(\epsilon\cdot\sigma)\dot{\Lambda}) \\ &= \frac{i}{2}\,\mathrm{Tr}(\underbrace{i\lambda\Lambda\sigma_{12}\Lambda^{-1}}_{S}(\epsilon\cdot\sigma)^{\bullet})\end{aligned}$$

Free lagrangian (cont.)

Angular momentum conservation

- Full variation now reads $\delta L_p = -i\,\mathrm{Tr}(k\,\epsilon\cdot\sigma) + \frac{i}{2}\,\mathrm{Tr}(S(\epsilon\cdot\sigma)^{\bullet})$
- Integrating by parts gives $\delta L_p = -i\,\mathrm{Tr}((z^a p^b + \frac{1}{2}S^{ab})^{\bullet}(\epsilon\cdot\sigma))$
- Conserved charge — total angular momentum:
  $M^{ab} = z^a p^b - z^b p^a + S^{ab}$
- If we expand $\dot{M} = 0$, multiply by p_a and recall that $p_a \dot{S}^{ab} = 0$, we'll obtain $p \parallel \dot{z}$
- Furthermore, then $p_a = m\dot{z}_a/\sqrt{-\dot{z}^2}$ and $\dot{S}^{ab} = 0$

Outline

Lagrangian with fields

- Analogous design of lagrangian

$$L = p_a e^a_\mu \dot{z}^\mu + i\frac{\lambda}{2}\,\mathrm{Tr}(\sigma_{12}\Lambda^{-1}\mathrm{D}_\tau\Lambda) + \text{field part}$$

- Covariant derivative — $(\mathrm{D}_\tau\Lambda)^a{}_b = \dot{\Lambda}^a{}_b + \dot{z}^\mu\omega^{ab}_\mu\Lambda_{cb}$
- Varying w.r.t. Λ is formally identical

$$\delta L = -i\,\mathrm{Tr}(J\,\epsilon\cdot\sigma) + \frac{i}{2}\,\mathrm{Tr}(S\mathrm{D}_\tau(\epsilon\cdot\sigma))$$

where $J^{ab} = e^a_\mu \dot{z}^\mu p^b$

Lagrangian with fields (cont.)

Spin precession equation

- Carrying on...

$$\begin{aligned} \delta L &= -i\,\mathrm{Tr}(J\,\epsilon\cdot\sigma) + \frac{i}{2}\,\mathrm{Tr}(S(\epsilon\cdot\sigma)^{\boldsymbol{\cdot}} + S[\dot{z}^\mu\omega_\mu, \epsilon\cdot\sigma]) \\ &\equiv -i\,\mathrm{Tr}(J\,\epsilon\cdot\sigma) + \frac{i}{2}\,\mathrm{Tr}(-\dot{S}\,\epsilon\cdot\sigma + S\dot{z}^\mu\omega_\mu\,\epsilon\cdot\sigma - \dot{z}^\mu\omega_\mu S\,\epsilon\cdot\sigma) \\ &= -i\,\mathrm{Tr}((J + \frac{1}{2}\mathrm{D}_\tau S)\epsilon\cdot\sigma) \end{aligned}$$

- EOM: $J^{ab} - J^{ba} + (\mathrm{D}_\tau S)^{ab} = 0$

Lagrangian with fields (cont.)

Varying w.r.t. z

This will hurt only for a while...

$$\begin{aligned}\delta L &= p_a e^a_\mu \delta\dot{z}^\mu + p_a \dot{z}^\lambda \delta z^\mu \partial_\mu e^a_\lambda + i\frac{\lambda}{2}\,\mathrm{Tr}(\sigma_{12}\Lambda^{-1}(\delta\dot{z}^\mu\omega_\mu + \dot{z}^\lambda\delta z^\mu\partial_\mu\omega_\lambda)\Lambda) \\ &= p_a e^a_\mu \delta\dot{z}^\mu + p_a \dot{z}^\lambda \delta z^\mu \partial_\mu e^a_\lambda + \frac{1}{2}\,\mathrm{Tr}(S(\delta\dot{z}^\mu\omega_\mu + \dot{z}^\lambda\delta z^\mu\partial_\mu\omega_\lambda)) \\ &\equiv -(p_a e^a_\mu)^{\boldsymbol{\cdot}}\delta z^\mu + p_a \dot{z}^\lambda \delta z^\mu \partial_\mu e^a_\lambda + \frac{1}{2}\,\mathrm{Tr}(-\dot{S}\omega_\mu - S\dot{\omega}_\mu + S\dot{z}^\lambda\partial_\mu\omega_\lambda)\delta z^\mu\end{aligned}$$

If $\tilde{J}^{ab} = J^{ab} - J^{ba}$ then $0 = \tilde{J} + \mathrm{D}_\tau S = \tilde{J} + \dot{S} + [\dot{z}^\lambda\omega_\lambda, S]$, so we expand $\dot{S} = -\tilde{J} - [\dot{z}^\lambda\omega_\lambda, S]$ into the following

$$\frac{\delta L}{\delta z^\mu} = -(p_a e^a_\mu)^{\boldsymbol{\cdot}} + p_a\dot{z}^\lambda\partial_\mu e^a_\lambda + \frac{1}{2}\,\mathrm{Tr}(\tilde{J}\omega_\mu + [\dot{z}^\lambda\omega_\lambda, S]\omega_\mu - S\dot{\omega}_\mu + S\dot{z}^\lambda\partial_\mu\omega_\lambda)$$

Lagrangian with fields (cont.)

Varying w.r.t. z (cont.)

We will now apply three simplifications at once:

- $\dot{\omega}_\mu = \frac{\partial \omega_\mu}{\partial z^\lambda} \frac{dz^\lambda}{d\tau} = \partial_\lambda \omega_\mu \dot{z}^\lambda$
- $\mathrm{Tr}([\dot{z}^\lambda \omega_\lambda, S]\omega_\mu) = \mathrm{Tr}(\dot{z}^\lambda S \omega_\mu \omega_\lambda - S \dot{z}^\lambda \omega_\lambda \omega_\mu) = \mathrm{Tr}(\dot{z}^\lambda S[\omega_\mu, \omega_\lambda])$
- $\mathrm{Tr}(\tilde{J}\omega_\mu) = (J_{ab} - J_{ba})\omega_\mu^{ba} = -2J_{ab}\omega_\mu^{ab} = -2e_a^\lambda \dot{z}_\lambda p_b \omega_\mu^{ab}$

$$\frac{\delta L}{\delta z^\mu} = -(p_a e_\mu^a)^{\boldsymbol{\cdot}} + p_a \dot{z}^\lambda \partial_\mu e_\lambda^a - e_a^\lambda \dot{z}_\lambda p_b \omega_\mu^{ab} + \frac{1}{2}\mathrm{Tr}(\dot{z}^\lambda S[\omega_\mu, \omega_\lambda] - \dot{z}^\lambda S \partial_\lambda \omega_\mu + \dot{z}^\lambda S \partial_\mu \omega_\lambda)$$

. . . and $\partial_\mu \omega_\lambda - \partial_\lambda \omega_\mu + [\omega_\mu, \omega_\lambda] = R_{\mu\lambda}$!

And we're done!

- Now we expand the first term

$$(p_a e^a_\mu)^{\cdot} = \dot{p}_a e^a_\mu + p_a \dot{e}^a_\mu = (\mathrm{D}_\tau p_a - \dot{z}^\lambda \omega_a{}^b{}_\lambda p_b) e^a_\mu + p_a \dot{z}^\lambda \partial_\lambda e^a_\mu$$

- After insertion we see

$$\frac{\delta L}{\delta z^\mu} = -e^a_\mu \mathrm{D}_\tau p_a + \frac{1}{2} \dot{z}^\lambda \operatorname{Tr}(S R_{\mu\lambda})$$
$$+ \underbrace{\dot{z}^\lambda \omega_a{}^b{}_\lambda p_b e^a_\mu - p_a \dot{z}^\lambda \partial_\lambda e^a_\mu + p_a \dot{z}^\lambda \partial_\mu e^a_\lambda - e^\lambda_a \dot{z}_\lambda p_b \omega^{ab}_\mu}_{p_a \dot{z}^\lambda \underbrace{(\partial_\mu e^a_\lambda - \partial_\lambda e^a_\mu + \omega^a_{\mu b} e^b_\lambda - \omega^a_{\lambda b} e^b_\mu)}_{T^a_{\mu\lambda}}}$$

Mathisson–Papapetrou equation (with torsion)

$$(\mathrm{D}_\tau p_a) e^a_\mu = p_a \dot{z}^\lambda T^a_{\mu\lambda} + \frac{1}{2} \dot{z}^\lambda \operatorname{Tr} S R_{\mu\nu}$$

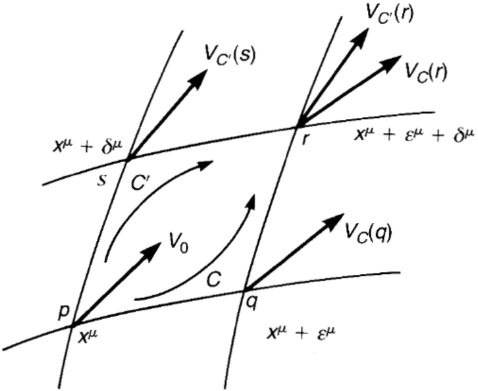
Outline

What's all this for?!

- E–C theory allows us to consider spinning particles
- No inherent reason for $T = 0$
- What is torsion itself, anyway?
- Simpliest physical realisation of translation — movement of a test particle by a fixed amount of affine parameter
- MP equation leads to EOMs for particles in contorted spacetime

What's all this for?!

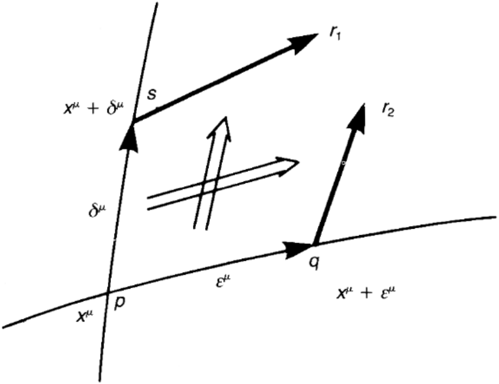
Curvature



Vector is rotated, when transported along a closed path

What's all this for?!

Torsion



Closed paths 'don't close', i.e. translation addition doesn't commute

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Selected references & further reading

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Myron Mathisson
Warsaw 1897 — Cambridge 1940