Gravity Theory — Homework #1

Paweł Laskoś-Grabowski

10 czerwca 2008

Verify that $\nabla_X(\nabla_Y Z^\mu) - \nabla_Y(\nabla_X Z^\mu) - \nabla_{[X,Y]}Z^\mu = R^\mu{}_{\alpha\nu\rho}Z^\alpha X^\nu Y^\rho$.

We'll use the following identities:

$$\begin{aligned}
\nabla_\mu X^\nu &= \partial_\mu X^\nu + \Gamma^\nu_{\mu\rho}X^\rho \\
\nabla_X Y^\mu &= X^\nu \nabla_\nu Y^\mu \\
[X,Y]^\nu &= X^\mu\partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu
\end{aligned}$$

Plugging all these into the lhs of the identity in question and expanding we obtain the following:

$$\begin{aligned}
\mathsf{LHS} &= X^\nu\nabla_\nu(Y^\rho\nabla_\rho Z^\mu) - Y^\nu\nabla_\nu(X^\rho\nabla_\rho Z^\mu) - [X,Y]^\nu\nabla_\nu Z^\mu = \\
&= X^\nu(\partial_\nu(Y^\rho\nabla_\rho Z^\mu) + \Gamma^\mu_{\sigma\nu}(Y^\rho\nabla_\rho Z^\sigma)) - Y^\nu(\partial_\nu(X^\rho\nabla_\rho Z^\mu) + \Gamma^\mu_{\sigma\nu}(X^\rho\nabla_\rho Z^\sigma)) - [X,Y]^\nu\nabla_\nu Z^\mu \\
&= X^\nu(\partial_\nu Y^\rho)\nabla_\rho Z^\mu + X^\nu Y^\rho\partial_\nu\nabla_\rho Z^\mu + X^\nu\Gamma^\mu_{\sigma\nu}Y^\rho\nabla_\rho Z^\sigma \\
&- Y^\nu(\partial_\nu X^\rho)\nabla_\rho Z^\mu - Y^\nu X^\rho\partial_\nu\nabla_\rho Z^\mu - Y^\nu\Gamma^\mu_{\sigma\nu}X^\rho\nabla_\rho Z^\sigma \\
&- (X^\rho\partial_\rho Y^\nu - Y^\rho\partial_\rho X^\nu)\nabla_\nu Z^\mu
\end{aligned}$$

Now we see that the last term cancels the first and fourth. We need to expand ∇'s in the remaining terms.

$$\begin{aligned}
\mathsf{LHS} &= X^\nu Y^\rho\partial_\nu\partial_\rho Z^\mu + X^\nu Y^\rho\partial_\nu(\Gamma^\mu_{\rho\sigma}Z^\sigma) + X^\nu\Gamma^\mu_{\sigma\nu}Y^\rho(\partial_\rho Z^\sigma + \Gamma^\sigma_{\rho\alpha}Z^\alpha) \\
&- Y^\nu X^\rho\partial_\nu\partial_\rho Z^\mu - Y^\nu X^\rho\partial_\nu(\Gamma^\mu_{\rho\sigma}Z^\sigma) - Y^\nu\Gamma^\mu_{\sigma\nu}X^\rho(\partial_\rho Z^\sigma + \Gamma^\sigma_{\rho\alpha}Z^\alpha)
\end{aligned}$$

First terms in each line cancel out, for $\partial_\nu, \partial_\rho$ commute.

$$\begin{aligned}
\mathsf{LHS} &= X^\nu Y^\rho(\partial_\nu\Gamma^\mu_{\rho\sigma})Z^\sigma + X^\nu Y^\rho\Gamma^\mu_{\rho\sigma}\partial_\nu Z^\sigma + X^\nu\Gamma^\mu_{\sigma\nu}Y^\rho\partial_\rho Z^\sigma + X^\nu\Gamma^\mu_{\sigma\nu}Y^\rho\Gamma^\sigma_{\rho\alpha}Z^\alpha \\
&- Y^\nu X^\rho(\partial_\nu\Gamma^\mu_{\rho\sigma})Z^\sigma - Y^\nu X^\rho\Gamma^\mu_{\rho\sigma}\partial_\nu Z^\sigma - Y^\nu\Gamma^\mu_{\sigma\nu}X^\rho\partial_\rho Z^\sigma - Y^\nu\Gamma^\mu_{\sigma\nu}X^\rho\Gamma^\sigma_{\rho\alpha}Z^\alpha
\end{aligned}$$

Second term in the first line cancels the third in the second line and vice versa. After renaming indices in the remaining terms, we obtain the desired result.

$$\begin{aligned}
\mathsf{LHS} &= (\partial_\nu\Gamma^\mu_{\rho\alpha})Z^\alpha X^\nu Y^\rho - (\partial_\rho\Gamma^\mu_{\nu\alpha})Z^\alpha X^\nu Y^\rho + \Gamma^\mu_{\sigma\nu}\Gamma^\sigma_{\rho\alpha}Z^\alpha X^\nu Y^\rho - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\alpha}Z^\alpha X^\nu Y^\rho \\
&= (\partial_\nu\Gamma^\mu_{\rho\alpha} - \partial_\rho\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\rho\alpha} - \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\alpha})Z^\alpha X^\nu Y^\rho = R^\mu{}_{\alpha\nu\rho}Z^\alpha X^\nu Y^\rho.
\end{aligned}$$

Verify that for geodesic $x(\tau)$, $\nabla_v v \equiv \nabla_{\dot{x}}\dot{x} = 0$.

By simply rewriting the expression we recover the geodesic equation.

$$\nabla_{\dot{x}}\dot{x} = \dot{x}^\mu(\partial_\mu\dot{x}^\nu + \Gamma^\nu_{\mu\rho}\dot{x}^\rho) \overset{\text{(chain rule)}}{=} \ddot{x}^\nu + \Gamma^\nu_{\mu\rho}\dot{x}^\mu\dot{x}^\rho = 0.$$