Classical field theory — exercise no. 1

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1 Content

Prove that for the action of the form $\mathcal{S} = \int_\Omega \mathrm{d}^4x\, \mathcal{L}(\partial_\mu \Phi^I, \Phi^I)$, the equations of motion are

$$\frac{\partial \mathcal{L}}{\partial \Phi^I} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^I)} = 0. \tag{1}$$

2 Solution

Let's examine the variation of a bivariate function expressed as a Taylor series (where we omit terms of non-linear order in $\delta x, \delta y$ by assumption that variations of the variables are small):

$$\delta f(x,y) = f(x+\delta x, y+\delta y) - f(x,y) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y. \tag{2}$$

Now let's substitute $f = \mathcal{L}$, $x = \partial_\mu \Phi$, $y = \Phi$ (we omit the index I for clarity, as it is insignificant for the calculations) and use the last result to express $\delta\mathcal{S}$:

$$\delta\mathcal{S} = \int_\Omega \mathrm{d}^4x\, \delta\mathcal{L} = \int_\Omega \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta(\partial_\mu \Phi) + \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi \right). \tag{3}$$

We'll prove that $\delta(\partial_\mu \Phi) = \partial_\mu(\delta\Phi)$ — in the following, let f be function mapping x to $x + \delta x$ (it is clear that $\partial_\mu f = 1$ as δx is arbitrary and thus does not depend on x):

$$\begin{aligned}\partial_\mu(\delta\Phi) &= \partial_\mu(\Phi(x+\delta x) - \Phi(x)) = \partial_\mu((\Phi \circ f)(x) - \Phi(x)) \\ &= (\partial_\mu \Phi)(f(x)) \times (\partial_\mu f)(x) - (\partial_\mu \Phi)(x) = (\partial_\mu \Phi)(x+\delta x) - (\partial_\mu \Phi)(x) \\ &= \delta(\partial_\mu \Phi). \end{aligned} \tag{4}$$

Let's express an instance of Leibniz law:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi \right) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu(\delta\Phi) + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \delta\Phi. \tag{5}$$

Under the integral in $\delta\mathcal{S}$, we add lhs and subtract rhs of the last result, and after one obvious (from (4)) term cancellation we obtain:

$$\begin{aligned}\delta\mathcal{S} &= \int_\Omega \mathrm{d}^4x \left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\delta\Phi + \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi \right) + \int_\Omega \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi \right) \\ &= \int_\Omega \mathrm{d}^4x \left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} + \frac{\partial \mathcal{L}}{\partial \Phi} \right) \delta\Phi + \int_{\partial\Omega} \mathrm{d}^3x\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi. \end{aligned} \tag{6}$$

The last term vanishes, as we assume $\delta\Phi = 0$ on the edge ($\partial\Omega$) of the space-time volume of integration. To obtain equations of motion we assume $\delta\mathcal{S} = 0$, and if $\delta\Phi$ is arbitrary, then by the fundamental lemma of calculus of variations we get that the term in parentheses is 0, which is the desired result.